Available online at www.sciencedirect.com

ScienceDirect

Journal of Geometry and Physics 57 (2007) 2419–2430

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Dijkgraaf–Witten invariants of surfaces and projective representations of groups

Vladimir Turaev*

IRMA, Université Louis Pasteur - C.N.R.S., 7 rue René Descartes, F-67084 Strasbourg, France
Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd Street, Bloomington, IN 47405, USA

Received 2 July 2007; accepted 11 August 2007
Available online 31 August 2007

Abstract

We compute the Dijkgraaf–Witten invariants of surfaces in terms of projective representations of groups. As an application we prove that the complex Dijkgraaf–Witten invariants of surfaces of positive genus are positive integers.

MSC: 57R56; 81T45; 20C25

Keywords: Surfaces; Dijkgraaf–Witten invariants; Projective representations

1. Introduction

Dijkgraaf and Witten [2] derived homotopy invariants of 3-manifolds from 3-dimensional cohomology classes of finite groups. Their construction provides examples of path integrals reduced to finite sums. It extends to arbitrary dimensions as follows. Fix a field F and let $F^* = F - \{0\}$ be the multiplicative group of non-zero elements of F. Fix a finite group G whose order $\#G$ is invertible in F. Pick an Eilenberg–MacLane CW-space X of type $K(G, 1)$ with base point $x \in X$. Consider a closed connected oriented topological manifold M of dimension $n \geq 1$ with base point $m_0 \in M$ and set $\pi = \pi_1(M, m_0)$. Observe that for any group homomorphism $\gamma : \pi \to G$, there is a mapping $f_\gamma : (M, m_0) \to (X, x)$ (unique up to homotopy) such that the induced homomorphism $(f_\gamma)_\# : \pi \to \pi_1(X, x) = G$ is equal to γ. The *Dijkgraaf–Witten invariant* $Z_\alpha(M) \in F$ determined by a cohomology class $\alpha \in H^n(G; F^*) = H^n(X; F^*)$ is defined by

$$Z_\alpha(M) = (\#G)^{-1} \sum_{\gamma \in \mathrm{Hom}(\pi, G)} \langle (f_\gamma)^*(\alpha), [M] \rangle. \tag{1.1}$$

Here $\mathrm{Hom}(\pi, G)$ is the (finite) set of all group homomorphisms $\pi \to G$ and

$$\langle (f_\gamma)^*(\alpha), [M] \rangle \in F^*$$

* Corresponding address: Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd Street, Bloomington, IN 47405, USA.
E-mail address: vturaev@yahoo.com.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.08.009

is the value of $(f_\gamma)^*(\alpha) \in H^n(M; F^*)$ on the fundamental class $[M] \in H_n(M; \mathbb{Z})$. The addition on the right-hand side of (1.1) is the addition in F. One may say that $Z_\alpha(M)$ counts the homomorphisms $\gamma : \pi \to G$ with weights $(\#G)^{-1}\langle (f_\gamma)^*(\alpha), [M]\rangle$. In particular, for $\gamma = 1$, we have $\langle (f_\gamma)^*(\alpha), [M]\rangle = 1_F$, where $1_F \in F$ is the unit of F. Thus, $\gamma = 1$ contributes the summand $(\#G)^{-1} \cdot 1_F$ to $Z_\alpha(M)$.

It is clear from the definitions that $Z_\alpha(M)$ depends neither on the choice of the base point $m_0 \in M$ nor on the choice of the Eilenberg–MacLane space X. Moreover, $Z_\alpha(M)$ depends only on α and the homotopy type of M. For example, if M is simply connected, that is if $\pi = \{1\}$, then $Z_\alpha(M) = (\#G)^{-1} \cdot 1_F$ for all α. If $G = \{1\}$, then $Z_\alpha(M) = 1_F$ for all M.

In this paper we study the case $n = 2$. We begin with elementary algebraic preliminaries. If $p \geq 0$ is the characteristic of the field F, then the formula $m \mapsto m \cdot 1_F$ for $m \in \mathbb{Z}/p\mathbb{Z}$, defines an isomorphism of $\mathbb{Z}/p\mathbb{Z}$ onto the ring $(\mathbb{Z}/p\mathbb{Z}) \cdot 1_F \subset F$ additively generated by 1_F. We identify $\mathbb{Z}/p\mathbb{Z}$ with $(\mathbb{Z}/p\mathbb{Z}) \cdot 1_F$ via this isomorphism. In particular, if $p = 0$, then $\mathbb{Z} = \mathbb{Z} \cdot 1_F \subset F$.

Theorem 1.1. *Let F be a field of characteristic $p \geq 0$ and $\alpha \in H^2(G; F^*)$. If $p = 0$, then $Z_\alpha(M) \in F$ is a positive integer for all closed connected oriented surfaces $M \neq S^2$. If $p > 0$, then $Z_\alpha(M) \in \mathbb{Z}/p\mathbb{Z} \subset F$ for all closed connected oriented surfaces M.*

The case of the sphere $M = S^2$ stays somewhat apart. Since S^2 is simply connected, $Z_\alpha(S^2) = (\#G)^{-1} \cdot 1_F$ for all α. If $p > 0$, then by the assumptions on G, the number $\#G$ is invertible in $\mathbb{Z}/p\mathbb{Z}$ and $Z_\alpha(S^2) \in \mathbb{Z}/p\mathbb{Z} \subset F$. If $p = 0$, then $Z_\alpha(S^2)$ is not an integer except for $G = \{1\}$.

Applying Theorem 1.1 to $F = \mathbb{C}$, we obtain that the complex number $Z_\alpha(M)$ is a positive integer for all surfaces $M \neq S^2$ and all $\alpha \in H^2(G; \mathbb{C}^*)$.

A curious feature of Theorem 1.1 is that its statement uses only classical notions of algebraic topology while its proof, given below, is based on ideas and techniques from quantum topology. I do not know how to prove this theorem using only the standard tools of algebraic topology.

Results similar to Theorem 1.1 are familiar in the study of 3-dimensional topological quantum field theories (TQFTs), where partition functions on surfaces compute dimensions of certain vector spaces associated with the surfaces. It would be interesting to give an interpretation of $Z_\alpha(M)$ as the dimension of a vector space naturally associated with M.

The proof of Theorem 1.1 is based on a Verlinde-type formula for $Z_\alpha(M)$ stated in terms of projective representations of G. Let, as above, F be a field (not necessarily algebraically closed). Fix a 2-cocycle $c : G \times G \to F^*$ so that for all $g_1, g_2, g_3 \in G$,

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_1, g_2g_3)c(g_2, g_3). \tag{1.2}$$

We will always assume that c is *normalized* in the sense that $c(g, 1) = c(1, g) = 1$ for all $g \in G$. (Any cohomology class in $H^2(G; F^*)$ can be represented by a normalized cocycle.) A mapping $\rho : G \to GL(W)$, where W is a finite-dimensional vector space over F, is called a *c-representation of* G if $\rho(1) = \mathrm{id}_W$ and $\rho(g_1g_2) = c(g_1, g_2)\rho(g_1)\rho(g_2)$ for any $g_1, g_2 \in G$. The dimension of W over F is denoted $\dim(\rho)$. Two c-representations $\rho : G \to GL(W)$ and $\rho' : G \to GL(W')$ are *equivalent* if there is an isomorphism of vector spaces $j : W \to W'$ such that $\rho'(g) = j\rho(g)j^{-1}$ for all $g \in G$. It is clear that $\dim(\rho)$ depends only on the equivalence class of ρ.

A c-representation $\rho : G \to GL(W)$ is *irreducible* if 0 and W are the only vector subspaces of W invariant under $\rho(G)$. It is obvious that a c-representation equivalent to an irreducible one is itself irreducible. The set of equivalence classes of irreducible c-representations of G is denoted $\widehat{G}_c$. We explain below that $\widehat{G}_c$ is a finite non-empty set.

The 2-cocycle c represents a cohomology class $[c] \in H^2(G; F^*)$. The theory of c-representations of G depends only on $[c]$. Indeed, any two normalized 2-cocycles $c, c' : G \times G \to F^*$ representing the same cohomology class satisfy

$$c(g_1, g_2) = c'(g_1, g_2)b(g_1)b(g_2)(b(g_1g_2))^{-1}$$

for all $g_1, g_2 \in G$ and a mapping $b : G \to F^*$ such that $b(1) = 1$. A c-representation ρ of G gives rise to a c'-representation ρ' of G by $\rho'(g) = b(g)\rho(g)$ for $g \in G$. This establishes a bijection between c-representations and c'-representations and induces a bijection $\widehat{G}_c \approx \widehat{G}_{c'}$.

Theorem 1.2. *Let F be an algebraically closed field such that $\#G$ is invertible in F. For any normalized 2-cocycle $c : G \times G \to F^*$ and any closed connected oriented surface M,*

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} \sum_{\rho \in \widehat{G}_c} (\dim \rho)^{\chi(M)} \cdot 1_F, \tag{1.3}$$

where $\chi(M)$ is the Euler characteristic of M.

It is known that the integer $\dim \rho$ divides $\#G$ in $\mathbb{Z}$ for any $\rho \in \widehat{G}_c$, see, for instance, [8, p. 296]. Therefore $\dim \rho$ is invertible in F so that the expression on the right-hand side of (1.3) is well-defined for all values of $\chi(M)$.

For $M = S^1 \times S^1$, Formula (1.3) gives $Z_{[c]}(S^1 \times S^1) = \#\widehat{G}_c (\operatorname{mod} p)$, where $p \geq 0$ is the characteristic of F. For $M = S^2$, Formula (1.3) can be rewritten as $\sum_{\rho \in \widehat{G}_c} (\dim \rho)^2 = \#G (\operatorname{mod} p)$. If $p = 0$, this gives

$$\sum_{\rho \in \widehat{G}_c} (\dim \rho)^2 = \#G. \tag{1.4}$$

We show below that (1.4) holds also when $p > 0$.

Denote M_k a closed connected oriented surface of genus $k \geq 0$. If $k \geq 1$, then $-\chi(M) = 2k - 2 \geq 0$ and

$$Z_{[c]}(M_k) = \sum_{\rho \in \widehat{G}_c} \left(\frac{\#G}{\dim \rho} \right)^{2k-2} \cdot 1_F$$

is a non-empty sum of positive integers times 1_F. This implies Theorem 1.1 for algebraically closed F. The general case of Theorem 1.1 follows by taking the algebraic closure of the ground field (this does not change the Dijkgraaf–Witten invariant).

For the trivial cocycle $c = 1$, we have $[c] = 0$ and $\widehat{G} = \widehat{G}_c$ is the set of all irreducible (finite-dimensional) linear representations of G over F considered up to linear equivalence. Clearly, $\langle (f_\gamma)^*([c]), [M] \rangle = 1 \in F^*$ for any closed connected oriented surface M and any homomorphism γ from $\pi = \pi_1(M)$ to G. Thus, $Z_0(M) = (\#G)^{-1} \# \operatorname{Hom}(\pi, G)$. Formula (1.3) gives

$$\# \operatorname{Hom}(\pi, G) = \#G \sum_{\rho \in \widehat{G}} \left(\frac{\#G}{\dim \rho} \right)^{-\chi(M)} (\operatorname{mod} p), \tag{1.5}$$

where $p \geq 0$ is the characteristic of F. For $M = S^1 \times S^1$ and $F = \mathbb{C}$, Formula (1.5) is due to Frobenius. For surfaces of higher genus, this formula was first pointed out by Mednykh [9], see [6] for a direct algebraic proof and [4, Section 5] for a proof based on topological field theory.

One can deduce further properties of $Z_{[c]}(M)$ from Theorem 1.2. Assume that F is an algebraically closed field of characteristic 0 and k is a positive integer. By Eq. (1.4), we have $(\#G)^{1/2} \geq \dim \rho \geq 1$ for all $\rho \in \widehat{G}_c$ and therefore

$$\#\widehat{G}_c (\#G)^{2k-2} \geq Z_{[c]}(M_k) \geq \#\widehat{G}_c (\#G)^{k-1}.$$

If $[c] \neq 0$, then $\dim \rho \geq 2$ for all $\rho \in \widehat{G}_c$ and we obtain a more precise estimate from above $\#\widehat{G}_c (\#G/2)^{2k-2} \geq Z_{[c]}(M_k)$.

If $\#G = q^N$, where q is a prime integer and $N \geq 1$, then $Z_{[c]}(M_k) \in \mathbb{Z}$ is divisible by $q^{N/2}$ for even N and by $q^{(N+1)/2}$ for odd N.

Formula (1.3) is suggested by the fact that the Dijkgraaf–Witten invariant (in an arbitrary dimension n) can be extended to an n-dimensional TQFT, see [2,13,4,3]. 2-dimensional TQFTs are known to arise from semisimple algebras, see [5]. We show that the 2-dimensional Dijkgraaf–Witten invariant $Z_{[c]}$ arises from the c-twisted group algebra of G. Then we split this algebra as a direct sum of matrix algebras numerated by the elements of $\widehat{G}_c$ and use a computation of the state sum invariants of surfaces derived from matrix algebras. Note that the proof of Eq. (1.3), detailed in Sections 2 and 3, actually does not use the notion of a TQFT. A version of these results for non-orientable surfaces is discussed in Section 4.

The author is indebted to Noah Snyder for a useful discussion of the non-orientable case.

2. State sum invariants of surfaces

We recall the state sum invariants of oriented surfaces associated with semisimple algebras, see [5].

Let $\mathcal{A}$ be a finite-dimensional algebra over a field F. For $a \in \mathcal{A}$, let $T(a) \in F$ be the trace of the F-linear homomorphism $\mathcal{A} \to \mathcal{A}$ sending any $x \in \mathcal{A}$ to ax. The resulting F-linear mapping $T : \mathcal{A} \to F$ is called the *trace homomorphism*. Clearly, $T(ab) = T(ba)$ for all $a, b \in \mathcal{A}$. Therefore the bilinear form $T^{(2)} : \mathcal{A} \otimes \mathcal{A} \to F$, defined by $T^{(2)}(a \otimes b) = T(ab)$ is symmetric. Assume that $\mathcal{A}$ is semisimple in the sense that the form $T^{(2)}$ is non-degenerate. We use the adjoint isomorphism $\operatorname{ad} T^{(2)}$ to identify $\mathcal{A}$ with the dual vector space $\operatorname{Hom}_F(\mathcal{A}, F)$. Using this identification and dualizing $T^{(2)} : \mathcal{A} \otimes \mathcal{A} \to F$, we obtain a homomorphism $F \to \mathcal{A} \otimes \mathcal{A}$. It sends $1 \in F$ to a vector

$$v = \sum_i v_i^1 \otimes v_i^2 \in \mathcal{A} \otimes \mathcal{A},$$

where i runs over a finite set of indices. The vector $v = v(\mathcal{A})$ is uniquely defined by the identity

$$T(ab) = \sum_i T(av_i^1)T(bv_i^2) \tag{2.1}$$

for all $a, b \in \mathcal{A}$. The vector v is symmetric in the sense that $\sum_i v_i^1 \otimes v_i^2 = \sum_i v_i^2 \otimes v_i^1$.

Consider a closed connected oriented surface M and fix a triangulation of M. We endow all 2-simplices of the triangulation of M with distinguished orientation induced by the one in M. A *flag* of M is a pair (a 2-simplex of the triangulation, an edge of this 2-simplex). Let $\{\mathcal{A}_f\}_f$ be a set of copies of $\mathcal{A}$ numerated by all flags f of M. Every edge e of (the triangulation of) M is incident to two 2-simplices Δ, Δ' of M. Let $v_e \in \mathcal{A}_{(\Delta,e)} \otimes \mathcal{A}_{(\Delta',e)}$ be a copy of $v \in \mathcal{A} \otimes \mathcal{A}$. The symmetry of v ensures that v_e is well-defined. Set $V = \otimes_e v_e \in \bigotimes_f \mathcal{A}_f$, where e runs over all edges of M and f runs over all flags of M.

We say that a trilinear form $U : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to F$ is *cyclically symmetric* if

$$U(a \otimes b \otimes c) = U(c \otimes a \otimes b)$$

for all $a, b, c \in \mathcal{A}$. Then U induces a homomorphism $\widetilde{U} : \bigotimes_f \mathcal{A}_f \to F$ as follows. Every 2-simplex Δ of M has three edges e_1, e_2, e_3 numerated so that following along the boundary of Δ in the direction determined by the distinguished orientation of Δ, one meets consecutively e_1, e_2, e_3. Since $\mathcal{A}_{(\Delta,e_i)} = \mathcal{A}$ for $i = 1, 2, 3$, the form U induces a trilinear form

$$U_\Delta : \mathcal{A}_{(\Delta,e_1)} \otimes \mathcal{A}_{(\Delta,e_2)} \otimes \mathcal{A}_{(\Delta,e_3)} \to F.$$

This form is cyclically symmetric and therefore independent of the numeration of the edges of Δ. The tensor product $\otimes_\Delta U_\Delta$ over all 2-simplices Δ of M is a homomorphism $\bigotimes_f \mathcal{A}_f \to F$ denoted $\widetilde{U}$.

The form $T^{(3)} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to F$ sending $a \otimes b \otimes c$ to $T(abc)$ for all $a, b, c \in \mathcal{A}$ is cyclically symmetric. Consider the induced homomorphism $\widetilde{T}^{(3)} : \bigotimes_f \mathcal{A}_f \to F$ and set $I_\mathcal{A}(M) = \widetilde{T}^{(3)}(V) \in F$. The key property of $I_\mathcal{A}(M)$ is the independence of the choice of triangulation of M. It is verified by checking the invariance of $I_\mathcal{A}(M)$ under the Pachner moves on the triangulations.

The direct product and the tensor product of two semisimple algebras are semisimple algebras. The invariant $I_\mathcal{A}(M)$ is additive with respect to direct products of algebras and multiplicative with respect to tensor products of algebras.

Lemma 2.1. *We have* $I_\mathcal{A}(S^2) = (\dim_F \mathcal{A}) \cdot 1_F$.

Proof. To compute $I_\mathcal{A}(S^2)$, we use the technique of skeletons (this technique will not be used elsewhere in this paper). A *skeleton of* a surface M is a finite graph embedded in M whose complement in M consists of open 2-disks. A triangulation of M gives rise to a skeleton of M whose vertices are the barycenters of the 2-simplices of the triangulation and whose edges are dual to the edges of the triangulation. One can rewrite the definition of $I_\mathcal{A}(M)$ in terms of state sums on skeletons, see [5,12]. Namely, one assigns $v = \sum_i v_i^1 \otimes v_i^2$ to each edge of the skeleton meaning that the index i is assigned to the edge, the element v_i^1 of $\mathcal{A}$ is assigned to one half-edge and v_i^2 to the other half-edge. For each vertex of the skeleton, one cyclically multiplies the elements of $\mathcal{A}$ assigned in this way to all incident half-edges and evaluates T on this product. These values of T are multiplied over all vertices of the skeleton and the results are summed over all the indices i sitting on the edges. The resulting sum is equal to $I_\mathcal{A}(M)$.

The 2-sphere S^2 has a skeleton $S^1 \subset S^2$ having one vertex and one edge. This gives $I_{\mathcal{A}}(S^2) = \sum_i T(v_i^1 v_i^2)$. To compute the latter expression, we can assume that the vectors $\{v_i^1\}_i$ in the expansion $v = \sum_i v_i^1 \otimes v_i^2$ form a basis of $\mathcal{A}$. Formula (2.1) implies that

$$T^{(2)}(a,b) = T^{(2)}\left(a, \sum_i T(bv_i^2)v_i^1\right),$$

where $T^{(2)}(a,b) = T(ab)$ for $a, b \in \mathcal{A}$. Since the bilinear form $T^{(2)}$ is non-degenerate, $b = \sum_i T(bv_i^2)v_i^1$ for all $b \in \mathcal{A}$. For $b = v_j^1$, this gives $v_j^1 = \sum_i T(v_j^1 v_i^2)v_i^1$. Since $\{v_i^1\}_i$ is a basis of $\mathcal{A}$, we have $T(v_j^1 v_i^2) = 1$ if $i = j$ and $T(v_j^1 v_i^2) = 0$ if $i \neq j$.

The trace of any F-linear homomorphism $f : \mathcal{A} \to \mathcal{A}$ can be expanded via the trace homomorphism $T : \mathcal{A} \to F$ as follows. Represent f by a matrix $(f_{i,j})$ over F in the basis $\{v_i^1\}_i$ so that $f(v_i^1) = \sum_j f_{i,j} v_j^1$ for all i. Then

$$\mathrm{Tr}(f) = \sum_i f_{i,i} = \sum_{i,j} f_{i,j} T(v_j^1 v_i^2) = \sum_i T(f(v_i^1)v_i^2).$$

Pick $a \in \mathcal{A}$ and consider the homomorphism $f_a : \mathcal{A} \to \mathcal{A}$ sending any $x \in \mathcal{A}$ to ax. By the previous formula,

$$T^{(2)}(a, 1_{\mathcal{A}}) = T(a) = \mathrm{Tr}(f_a) = T\left(\sum_i a v_i^1 v_i^2\right) = T^{(2)}\left(a, \sum_i v_i^1 v_i^2\right),$$

where $1_{\mathcal{A}}$ is the unit of $\mathcal{A}$. The non-degeneracy of $T^{(2)}$ implies that $\sum_i v_i^1 v_i^2 = 1_{\mathcal{A}}$. Thus,

$$I_{\mathcal{A}}(S^2) = T\left(\sum_i v_i^1 v_i^2\right) = T(1_{\mathcal{A}}) = (\dim_F \mathcal{A}) \cdot 1_F. \quad \square$$

Example. Let $\mathcal{A} = \mathrm{Mat}_d(F)$ be the algebra of $(d \times d)$-matrices over F with $d \geq 1$. This algebra is semisimple if and only if d is invertible in F and then $I_{\mathcal{A}}(M) = d^{\chi(M)} \cdot 1_F$ for any closed connected oriented surface M, see [11, Theorem 4.2]. For $d = 1$, we obtain $I_F(M) = 1_F$ for all M.

3. Invariants derived from twisted group algebras

Let G be a group and $F[G]$ be the vector space over a field F with basis G. A normalized 2-cocycle $c : G \times G \to F^*$ gives rise to a multiplication law $\cdot$ on $F[G]$ by $g_1 \cdot g_2 = c(g_1, g_2)g_1g_2$, where g_1, g_2 run over G and $g_1g_2 \in G$ is the product in G. The vector space $F[G]$ with this multiplication is an associative algebra and the neutral element $1 \in G \subset F[G]$ is its unit. This algebra is called the *twisted group algebra of* G and denoted $A^{(c)}$. It is easy to check that the isomorphism type of $A^{(c)}$ depends only on the cohomology class $[c] \in H^2(G; F^*)$.

From now on, G is a finite group whose order $\#G$ is invertible in F. The algebra $A^{(c)}$ is $(\#G)$-dimensional and the trace homomorphism $T : A^{(c)} \to F$ defined in Section 2 sends $1 \in G$ to $\#G$ and sends all other basis vectors of $A^{(c)}$ to zero. The associated bilinear form $T^{(2)} : A^{(c)} \otimes A^{(c)} \to F$ sends a pair of basis vectors $g_1, g_2 \in G$ to $\#G$ if $g_2 = g_1^{-1}$ and to 0 otherwise. This form is non-degenerate and so the algebra $A^{(c)}$ is semisimple. The following theorem shows that the state sum invariant of surfaces $I_{A^{(c)}}$ is equivalent to the Dijkgraaf–Witten invariant derived from $[c] \in H^2(G; F^*)$.

Theorem 3.1. *For any normalized 2-cocycle $c : G \times G \to F^*$ on G and any closed connected oriented surface M,*

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} I_{A^{(c)}}(M).$$

Proof. Fix a triangulation of M and let k_0, k_1, k_2 be respectively the number of vertices, edges, and 2-simplices of this triangulation. By an *oriented edge* of M we mean an edge of (the triangulation of) M endowed with an arbitrary orientation. For an oriented edge e of M, the same edge with opposite orientation is denoted $-e$. A *labeling* of M is a mapping ℓ from the set of oriented edges of M to G such that $\ell(-e) = (\ell(e))^{-1}$ for all oriented edges e of M. A labeling ℓ of M is *admissible* if $\ell(e_1)\ell(e_2)\ell(e_3) = 1$ for any three consecutive oriented edges e_1, e_2, e_3 forming the boundary of a 2-simplex of (the triangulation of) M. Denote the set of labelings of M by $L(M)$ and denote its subset formed by the admissible labelings by $L_a(M)$.

Given a labeling $\ell \in L(M)$, we assign to any path p in M formed by consecutive oriented edges $e_1, \ldots, e_N$ the product $\ell(p) = \ell(e_1)\ell(e_2)\cdots\ell(e_N) \in G$. For admissible ℓ, this product is a homotopy invariant of p: if two paths p, p' have the same endpoints and are homotopic (relative to the endpoints), then $\ell(p) = \ell(p')$.

Fix a base vertex $m_0 \in M$ and set $\pi = \pi_1(M, m_0)$. For any $\ell \in L_a(M)$, applying the mapping $p \mapsto \ell(p)$ to the loops in M based at m_0, we obtain a group homomorphism $\pi \to G$ denoted $\Gamma(\ell)$. The formula $\ell \mapsto \Gamma(\ell)$ defines a mapping $\Gamma : L_a(M) \to \mathrm{Hom}(\pi, G)$.

We claim that the pre-image $\Gamma^{-1}(\gamma)$ of any $\gamma \in \mathrm{Hom}(\pi, G)$ consists of $(\#G)^{k_0-1}$ admissible labelings. To see this, fix a spanning tree $R \subset M$ formed by all k_0 vertices and $k_0 - 1$ edges of M; here we use that M is connected. For every vertex m of M, there is a (unique up to homotopy) path p_m in R formed by oriented edges of R and leading from m_0 to m. Any oriented edge e of M not lying in R determines a loop $p_{s_e}e(p_{t_e})^{-1}$, where s_e and t_e are the initial and the terminal endpoints of e, respectively. The homotopy classes of such loops corresponding to all oriented edges e of M not lying in R generate the fundamental group π. Therefore the pre-image of $\gamma \in \mathrm{Hom}(\pi, G)$ consists of the labelings $\ell \in L_a(M)$ such that $\ell(p_{s_e}e(p_{t_e})^{-1}) = \gamma(p_{s_e}e(p_{t_e})^{-1})$ for all e as above. This equality may be rewritten as

$$\ell(e) = (\ell(p_{s_e}))^{-1}\gamma(p_{s_e}e(p_{t_e})^{-1})\ell(p_{t_e}). \tag{3.1}$$

Therefore to specify $\ell \in \Gamma^{-1}(\gamma)$, we can assign arbitrary labels to the $k_0 - 1$ edges of R oriented away from m_0, the inverse labels to the same edges oriented towards m_0, and the labels determined from Eq. (3.1) to the oriented edges of M not lying in R. The resulting labeling is necessarily admissible. Hence, $\#\,\Gamma^{-1}(\gamma) = (\#G)^{k_0-1}$.

Formula (1.1) and the results above imply that

$$Z_{[c]}(M) = (\#G)^{-k_0} \sum_{\ell \in L_a(M)} \langle (f_{\Gamma(\ell)})^*([c]), [M] \rangle \in F,$$

where $f_{\Gamma(\ell)}$ is a mapping from the pair (M, m_0) to the pair (an Eilenberg–MacLane space X of type $K(G, 1)$, a base point $x \in X$) such that the induced homomorphism of fundamental groups is equal to $\Gamma(\ell) : \pi \to G$. Choosing in the role of X the canonical realization of the Eilenberg–MacLane space $K(G, 1)$ associated with the standard resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$ (see, for instance, [1]), we can compute $\langle (f_{\Gamma(\ell)})^*([c]), [M] \rangle$ as follows. Fix a total order $<$ on the set of all vertices of M. A 2-simplex Δ of M has three vertices A, B, C with $A < B < C$. Set $\varepsilon_\Delta = +1$ if the distinguished orientation of Δ (induced by the one on M) induces the direction from A to B on the edge $AB \subset \partial\Delta$ and set $\varepsilon_\Delta = -1$ otherwise. Let $\ell_1^\Delta = \ell(AB)$ and $\ell_2^\Delta = \ell(BC)$ be the labels of the edges AB, BC oriented from A to B and from B to C, respectively. Then

$$\langle (f_{\Gamma(\ell)})^*([c]), [M] \rangle = \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta)^{\varepsilon_\Delta} \in F^*,$$

where Δ runs over all 2-simplices of M. Hence,

$$Z_{[c]}(M) = (\#G)^{-k_0} \sum_{\ell \in L_a(M)} \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta)^{\varepsilon_\Delta}. \tag{3.2}$$

We now compute $I_{\mathcal{A}}(M) \in F$ for $\mathcal{A} = A^{(c)}$. First, with each labeling $\ell \in L(M)$ we associate an element $\langle c, \ell \rangle$ of F^*. Observe that $c(g, g^{-1}) = c(g^{-1}, g)$ for all $g \in G$ (this is obtained from Eq. (1.2) by the substitution $g_1 = g_3 = g$ and $g_2 = g^{-1}$). Therefore for any oriented edge e of M, the expression $c(\ell(e), \ell(-e)) = c(\ell(-e), \ell(e)) \in F^*$ does not depend on the orientation of e and may be associated with the underlying unoriented edge. Set

$$\langle c, \ell \rangle = \prod_e c(\ell(e), \ell(-e)) \in F^*,$$

where e runs over all non-oriented edges of M.

Let as above $\{\mathcal{A}_f\}_f$ be a set of copies of $\mathcal{A}$ numerated by all flags f of M. With a labeling $\ell \in L(M)$ we associate a vector $V(\ell) \in \otimes_f \mathcal{A}_f$ as follows. For a flag f formed by a 2-simplex Δ and its edge e, the distinguished orientation of Δ induces an orientation of e. Let $\ell(f) \in G \subset \mathcal{A} = \mathcal{A}_f$ be the value of ℓ on this oriented edge. Set $V(\ell) = \otimes_f \ell(f)$.

Recall the homomorphisms $T : \mathcal{A} \to F$, $T^{(3)} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to F$, and the vector $v \in \mathcal{A} \otimes \mathcal{A}$ introduced in Section 2. It is easy to compute that

$$v = (\#G)^{-1} \sum_{g \in G} (c(g, g^{-1}))^{-1} g \otimes g^{-1}.$$

Therefore the vector $V = \otimes_e v_e \in \otimes_f \mathcal{A}_f$ is computed by

$$V = (\#G)^{-k_1} \sum_{\ell \in L(M)} \langle c, \ell \rangle^{-1} V(\ell).$$

For any $g_1, g_2, g_3 \in G$, we have $T^{(3)}(g_1 \otimes g_2 \otimes g_3) = 0$ if $g_1 g_2 g_3 \neq 1$ and

$$T^{(3)}(g_1 \otimes g_2 \otimes g_3) = T(g_1 g_2 g_3) = \#G c(g_1, g_2) c(g_1 g_2, g_3)$$

if $g_1 g_2 g_3 = 1$. Consider the homomorphism $U = (\#G)^{-1} T^{(3)} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to F$ sending $g_1 \otimes g_2 \otimes g_3$ to 0 if $g_1 g_2 g_3 \neq 1$ and to $c(g_1, g_2) c(g_1 g_2, g_3)$ if $g_1 g_2 g_3 = 1$. The cyclic symmetry of $T^{(3)}$ implies that U is cyclically symmetric. Then

$$I_{\mathcal{A}}(M) = \widetilde{T}^{(3)}(V) = (\#G)^{k_2} \widetilde{U}(V) = (\#G)^{k_2 - k_1} \sum_{\ell \in L(M)} \langle c, \ell \rangle^{-1} \widetilde{U}(V(\ell)).$$

It is clear that $k_2 - k_1 = \chi(M) - k_0$ and $\widetilde{U}(V(\ell)) = 0$ for non-admissible ℓ. Hence

$$I_{\mathcal{A}}(M) = (\#G)^{\chi(M) - k_0} \sum_{\ell \in L_a(M)} \langle c, \ell \rangle^{-1} \widetilde{U}(V(\ell)). \tag{3.3}$$

To compute $\widetilde{U}(V(\ell))$ for $\ell \in L_a(M)$, we use the total order $<$ on the set of vertices of M. With a 2-simplex $\Delta = ABC$ of M with $A < B < C$ we associated above a sign $\varepsilon_\Delta = \pm 1$ and two labels $g_1 = \ell_1^\Delta = \ell(AB) \in G$ and $g_2 = \ell_2^\Delta = \ell(BC) \in G$. Set also $g_3 = \ell_3^\Delta = \ell(CA) \in G$. The admissibility of ℓ implies that $g_1 g_2 g_3 = 1$. The 2-simplex Δ gives rise to the flags (Δ, AB), (Δ, BC), and (Δ, CA). These flags contribute to $V(\ell)$ the tensor factor

$$(g_1)^{\varepsilon_\Delta} \otimes (g_2)^{\varepsilon_\Delta} \otimes (g_3)^{\varepsilon_\Delta} \in \mathcal{A}_{(\Delta, AB)} \otimes \mathcal{A}_{(\Delta, BC)} \otimes \mathcal{A}_{(\Delta, CA)}.$$

Recall the trilinear form $U_\Delta : \mathcal{A}_{(\Delta, AB)} \otimes \mathcal{A}_{(\Delta, BC)} \otimes \mathcal{A}_{(\Delta, CA)} \to F$ introduced in Section 2. If $\varepsilon_\Delta = +1$, then

$$\begin{aligned} U_\Delta((g_1)^{\varepsilon_\Delta} \otimes (g_2)^{\varepsilon_\Delta} \otimes (g_3)^{\varepsilon_\Delta}) &= U(g_1 \otimes g_2 \otimes g_3) = c(g_1, g_2) c(g_1 g_2, g_3) \\ &= c(g_1, g_2) c(g_3^{-1}, g_3) = c(g_1, g_2) c(g_3, g_3^{-1}). \end{aligned}$$

If $\varepsilon_\Delta = -1$, then

$$\begin{aligned} U_\Delta((g_1)^{\varepsilon_\Delta} \otimes (g_2)^{\varepsilon_\Delta} \otimes (g_3)^{\varepsilon_\Delta}) &= U_\Delta(g_1^{-1} \otimes g_2^{-1} \otimes g_3^{-1}) = U(g_3^{-1} \otimes g_2^{-1} \otimes g_1^{-1}) \\ &= c(g_3^{-1}, g_2^{-1}) c(g_3^{-1} g_2^{-1}, g_1^{-1}) = c(g_1 g_2, g_2^{-1}) c(g_1, g_1^{-1}) \\ &= (c(g_1, g_2))^{-1} c(g_1, g_1^{-1}) c(g_2, g_2^{-1}). \end{aligned}$$

The last equality follows from (1.2), where we set $g_3 = g_2^{-1}$. In both cases

$$U_\Delta((g_1)^{\varepsilon_\Delta} \otimes (g_2)^{\varepsilon_\Delta} \otimes (g_3)^{\varepsilon_\Delta}) = c(g_1, g_2)^{\varepsilon_\Delta} u_\Delta,$$

where $u_\Delta = c(g_3, g_3^{-1})$ if $\varepsilon_\Delta = +1$ and $u_\Delta = c(g_1, g_1^{-1}) c(g_2, g_2^{-1})$ if $\varepsilon_\Delta = -1$. We conclude that

$$\widetilde{U}(V(\ell)) = \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta)^{\varepsilon_\Delta} u_\Delta, \tag{3.4}$$

where Δ runs over all 2-simplices of M.

We claim that $\prod_\Delta u_\Delta = \langle c, \ell \rangle$. Note that the product $\prod_\Delta u_\Delta$ expands as a product of the expressions $c(\ell(e), \ell(-e))$ associated with edges e of M. We show that every edge $e = AB$ of M with $A < B$ contributes exactly one such expression. Set $g = \ell(AB) \in G$. The edge AB is incident to two 2-simplices $\Delta = ABC$ and $\Delta' = ABC'$ of M whose distinguished orientations induce on $AB = \partial\Delta \cap \partial\Delta'$ the directions from B to A and from A to B, respectively. If $B < C$, then $\varepsilon_\Delta = -1$, $g = \ell_1^\Delta$, and AB contributes the factor $c(g, g^{-1})$ to u_Δ. If $C < A$, then $\varepsilon_\Delta = -1$, $g = \ell_2^\Delta$, and AB contributes the factor $c(g, g^{-1})$ to u_Δ. Finally, if $A < C < B$, then $\varepsilon_\Delta = +1$, $g = \ell_3^\Delta$, and $u_\Delta = c(g, g^{-1})$. A similar computation shows that AB contributes no factors to $u_{\Delta'}$. Therefore

$$\prod_\Delta u_\Delta = \prod_e c(\ell(e), \ell(-e)) = \langle c, \ell\rangle.$$

Substituting this in (3.4), we obtain

$$\widetilde{U}(V(\ell)) = \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta)^{\varepsilon_\Delta} \times \langle c, \ell\rangle.$$

Formula (3.3) yields

$$I_{\mathcal{A}}(M) = (\#G)^{\chi(M)-k_0} \sum_{\ell\in L_a(M)} \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta)^{\varepsilon_\Delta}.$$

Comparing with (3.2), we obtain the claim of the theorem. □

Since the algebra $A^{(c)}$ is finite-dimensional and semisimple, the isomorphism classes of simple $A^{(c)}$-modules form a finite non-empty set Λ (an $A^{(c)}$-module is *simple* if its only $A^{(c)}$-submodules are itself and zero). Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be representatives of these isomorphism classes. Then

$$A^{(c)} \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Mat}_{d_\lambda}(D_\lambda) = \bigoplus_{\lambda\in\Lambda} D_\lambda \otimes_F \mathrm{Mat}_{d_\lambda}(F), \tag{3.5}$$

where $D_\lambda = \mathrm{End}_{A^{(c)}}(V_\lambda)$ is a division F-algebra (i.e., an F-algebra in which all non-zero elements are invertible), d_λ is the dimension of V_λ as a D_λ-module, and $\mathrm{Mat}_d(D)$ with $d \geq 1$ is the algebra of $(d \times d)$-matrices over the ring D. The integer d_λ is invertible in F for all λ, because the algebra $\mathrm{Mat}_{d_\lambda}(D_\lambda)$ is a direct summand of $A^{(c)}$ and is therefore semisimple. Theorem 3.1 implies that (under the conditions of this theorem)

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} \sum_{\lambda\in\Lambda} I_{D_\lambda}(M) d_\lambda^{\chi(M)}. \tag{3.6}$$

We can check (3.6) directly for $M = S^2$. Indeed, $Z_{[c]}(S^2) = (\#G)^{-1} \cdot 1_F$ and by Lemma 2.1, $I_{D_\lambda}(S^2) = (\dim_F D_\lambda) \cdot 1_F$ for all $\lambda \in \Lambda$. Therefore (3.6) for $M = S^2$ follows from the equality

$$\#G = \sum_{\lambda\in\Lambda} (\dim_F D_\lambda) d_\lambda^2 \tag{3.7}$$

which is a consequence of (3.5).

Formulas (3.6) and (3.7) simplify in the case where F is algebraically closed. Then $D_\lambda = F$ for all $\lambda \in \Lambda$ and we obtain

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} \sum_{\lambda\in\Lambda} d_\lambda^{\chi(M)} \cdot 1_F \tag{3.8}$$

and

$$\#G = \sum_{\lambda\in\Lambda} d_\lambda^2. \tag{3.9}$$

In particular, $Z_{[c]}(S^1 \times S^1) = \#\Lambda \cdot 1_F$. The number $\#\Lambda$, that is the number of isomorphism classes of simple $A^{(c)}$-modules, can be computed in terms of the so-called c-regular classes of G, see [8, pp. 107–118]. An element $g \in G$ is *c-regular* if $c(g, h) = c(h, g)$ for all $h \in G$ such that $gh = hg$. The set of c-regular elements of G depends only on the cohomology class $[c] \in H^2(G; F^*)$ and is invariant under conjugation in G. Since F is algebraically closed (and as always in this paper, $\#G$ is invertible in F), $\#\Lambda = r(G; c)$, where $r(G; c)$ is the number of conjugacy classes of c-regular elements of G, see [8, p. 117].

Proof of Theorem 1.2. Any c-representation of G in the sense of Section 1 extends by linearity to an action of $A^{(c)}$ on the corresponding vector space. This gives a bijective correspondence between c-representations of G and $A^{(c)}$-modules of finite dimension over F. This correspondence transforms equivalent representations to isomorphic $A^{(c)}$-modules and irreducible representations to simple modules. This allows us to rewrite all the statements of this section in terms of the c-representations of G. In particular, the set $\widehat{G}_c$ of equivalence classes of irreducible c-representations of G is finite and non-empty. Now, Formula (3.8) directly implies (1.3). Note also that Formula (3.9) implies (1.4).

Remark 3.2. Formula (1.5) generalizes to surfaces with boundary as follows (see [10,6]). Let π be the fundamental group of a compact connected oriented surface M whose boundary consists of $k \geq 1$ circles and let $x_1, \ldots, x_k$ be the conjugacy classes in π represented by the components of ∂M. For any $g_1, \ldots, g_k \in G$, the number of homomorphisms $\varphi : \pi \to G$ such that $\varphi(x_i)$ is conjugate to g_i in G for all $i = 1, \ldots, k$ is equal to

$$\#G \sum_{\rho \in \widehat{G}} \left(\frac{\#G}{\dim \rho} \right)^{-\chi(M)} \prod_{i=1}^{k} \chi_\rho(g_i) (\mathrm{mod}\, p),$$

where $\chi_\rho : G \to F$ is the characteristic of ρ and p is the characteristic of F. It would be interesting to give a similar generalization of (1.3).

4. The non-orientable case

A non-oriented version of the Dijkgraaf–Witten invariant in dimension $n \geq 1$ can be defined as follows. Let G be a finite group and $\alpha \in H^n(G; \mathbb{Z}/2\mathbb{Z})$. For a closed connected n-dimensional topological manifold M with fundamental group π, set

$$Z_\alpha(M) = (\#G)^{-1} \sum_{\gamma \in \mathrm{Hom}(\pi, G)} (-1)^{\langle (f_\gamma)^*(\alpha), [M] \rangle} \in (\#G)^{-1}\, \mathbb{Z},$$

where $f_\gamma : M \to K(G, 1)$ is as in Section 1 and $\langle (f_\gamma)^*(\alpha), [M] \rangle \in \mathbb{Z}/2\mathbb{Z}$ is the value of $(f_\gamma)^*(\alpha) \in H^n(M; \mathbb{Z}/2\mathbb{Z})$ on the fundamental class $[M] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$. For orientable M, this is a special case of the definition given in Section 1. We therefore restrict ourselves to non-orientable M. From now on, $n = 2$.

We formulate a Verlinde-type formula for $Z_\alpha(M)$. Fix a normalized 2-cocycle c on G with values in the cyclic group of order two $\{\pm 1\}$. Fix a field F such that $\#G$ is invertible in F. An irreducible c-representation $\rho : G \to GL(W)$, where W is a finite-dimensional vector space over F, is *self-dual* if there is a non-degenerate bilinear form $\langle , \rangle : W \times W \to F$ such that $\langle \rho(g)(u), \rho(g)(v) \rangle = \langle u, v \rangle$ for all $g \in G$ and $u, v \in W$. The irreducibility of ρ easily implies that the form $\langle , \rangle$ has to be either symmetric or skew-symmetric. Set $\varepsilon_\rho = 1 \in F$ in the former case and $\varepsilon_\rho = -1 \in F$ in the latter case. For any non-self-dual irreducible c-representation ρ, set $\varepsilon_\rho = 0 \in F$. The formula $\rho \mapsto \varepsilon_\rho$ yields a well-defined function $\widehat{G}_c \to \{-1, 0, 1\} \subset F$. This function plays the role of the Frobenius–Schur indicator in the theory of representations over $\mathbb{C}$.

Composing $c : G \times G \to \{\pm 1\}$ with the isomorphism $\{\pm 1\} \approx \mathbb{Z}/2\mathbb{Z}$, we obtain a $(\mathbb{Z}/2\mathbb{Z})$-valued 2-cocycle on G. Its cohomology class in $H^2(G; \mathbb{Z}/2\mathbb{Z})$ is denoted $[c]$.

Theorem 4.1. *If F is an algebraically closed field, then for any normalized 2-cocycle $c : G \times G \to \{\pm 1\}$ and any closed connected non-orientable surface M,*

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} \sum_{\rho \in \widehat{G}_c} (\varepsilon_\rho \dim \rho)^{\chi(M)}. \tag{4.1}$$

Note that $\chi(M) \leq 0$ for all closed connected non-orientable surfaces M distinct from the real projective plane P^2.

Corollary 4.2. *Let F be a field of characteristic $p \geq 0$ and $\alpha \in H^2(G; \mathbb{Z}/2\mathbb{Z})$. If $p = 0$, then $Z_\alpha(M) \in F$ is a non-negative integer for all closed connected non-orientable surfaces $M \neq P^2$. If $p > 0$, then $Z_\alpha(M) \in \mathbb{Z}/p\mathbb{Z} \subset F$ for all closed connected non-orientable surfaces M.*

The proof of Theorem 4.1 uses state sum invariants of non-oriented surfaces introduced by Karimipour and Mostafazadeh [7], see also [11]. These invariants are derived from the so-called $*$-algebras. A $*$-*algebra* over a field F (not necessarily algebraically closed) is a finite-dimensional algebra $\mathcal{A}$ over F endowed with an F-linear involution $\mathcal{A} \to \mathcal{A}, a \mapsto a^*$ such that $(ab)^* = b^* a^*$ for all $a, b \in \mathcal{A}$ and $T(a^*) = T(a)$ for all $a \in \mathcal{A}$, where $T : \mathcal{A} \to F$ is the trace homomorphism defined in Section 2. Note that for any $a, b \in \mathcal{A}$,

$$T(a^* b) = T((a^* b)^*) = T(b^* a) = T(a b^*).$$

A $*$-algebra $\mathcal{A}$ is *semisimple* if the bilinear form $T^{(2)} : \mathcal{A}\otimes\mathcal{A} \to F$, defined by $a\otimes b \mapsto T(ab)$, is non-degenerate. Consider the vector $v = \sum_i v_i^1 \otimes v_i^2 \in \mathcal{A}\otimes\mathcal{A}$ satisfying (2.1) and note that

$$(\mathrm{id}_{\mathcal{A}} \otimes *)(v) = (* \otimes \mathrm{id})(v). \tag{4.2}$$

Indeed, for any $a, b \in \mathcal{A}$,

$$\sum_i T(a(v_i^1)^*)T(bv_i^2) = \sum_i T(a^*v_i^1)T(bv_i^2) = T(a^*b) = T(ab^*)$$
$$= \sum_i T(av_i^1)T(b^*v_i^2) = \sum_i T(av_i^1)T(b(v_i^2)^*).$$

Now, the non-degeneracy of $T^{(2)}$ implies that $\sum_i (v_i^1)^* \otimes v_i^2 = \sum_i v_i^1 \otimes (v_i^2)^*$.

A semisimple $*$-algebra $(\mathcal{A}, *)$ over F gives rise to an invariant of a closed connected (non-oriented) surface M as follows (cf. [7,11]). Fix a triangulation of M and an arbitrary orientation on its 2-simplices. Then proceed as in Section 2 with one change: the vector v_e assigned to an edge e is v if the orientations of two 2-simplices adjacent to e induce opposite orientations on e and is $(\mathrm{id}_{\mathcal{A}} \otimes *)(v) = (* \otimes \mathrm{id}_{\mathcal{A}})(v)$ otherwise. Then $I_{(\mathcal{A},*)}(M) = \widetilde{T}^{(3)}(V) \in F$ is a topological invariant of M, where $V = \otimes_e v_e$. That $I_{(\mathcal{A},*)}(M)$ is preserved when the orientation on a 2-simplex is reversed follows from the formula $T(abc) = T((abc)^*) = T(c^*b^*a^*)$ for $a, b, c \in \mathcal{A}$. For orientable M, we have $I_{(\mathcal{A},*)}(M) = I_{\mathcal{A}}(M)$, where $I_{\mathcal{A}}(M)$ is the invariant of Section 2 computed for an arbitrary orientation of M.

The following example was communicated to the author by Snyder, cf. [11, Theorem 4.2]. Let $\mathcal{A} = \mathrm{Mat}_d(F)$ with $d \geq 1$ invertible in F and let $*$ be the involution in $\mathcal{A}$ defined by $a^* = Q^{-1}a^{\mathrm{Tr}}Q$, where $a \in \mathcal{A}$ and $Q \in \mathrm{Mat}_d(F)$ is an invertible matrix such that $Q^{\mathrm{Tr}} = \varepsilon Q$ for $\varepsilon = \pm 1$. Then the pair $(\mathcal{A}, *)$ is a semisimple $*$-algebra and $I_{(\mathcal{A},*)}(M) = (\varepsilon d)^{\chi(M)} \cdot 1_F$.

Let again $c : G \times G \to \{\pm 1\}$ be a normalized 2-cocycle. We define a multiplication $\cdot$ on the vector space $F[G]$ by $g_1 \cdot g_2 = c(g_1, g_2)g_1g_2$ for any $g_1, g_2 \in G$. This turns $F[G]$ into an associative unital algebra $A^{(c)}$ with involution $*$ defined by $g^* = c(g, g^{-1})g^{-1}$ for $g \in G$. The pair $(A^{(c)}, *)$ is a semisimple $*$-algebra. The only non-obvious condition is the equality $(ab)^* = b^*a^*$ for $a, b \in A^{(c)}$. It suffices to check this equality for $a, b \in G$. It is equivalent then to the five-term identity

$$c(ab, (ab)^{-1}) = c(a, a^{-1})c(b, b^{-1})c(a, b)c(b^{-1}, a^{-1}).$$

To check this identity, we substitute $g_1 = a, g_2 = b, g_3 = b^{-1}$ in (1.2) and obtain $c(ab, b^{-1}) = c(a, b)c(b, b^{-1})$. Then set $g_1 = ab, g_2 = b^{-1}, g_3 = a^{-1}$ in (1.2) and substitute $c(ab, b^{-1}) = c(a, b)c(b, b^{-1})$ in the resulting formula. This yields a formula equivalent to the five-term identity.

Theorem 4.3. *For any closed connected surface M,*

$$Z_{[c]}(M) = (\#G)^{-\chi(M)} I_{(A^{(c)},*)}(M).$$

Proof. The proof is analogous to the proof of Theorem 3.1 and we only indicate the main changes. One begins by fixing a triangulation of M and a total order on the set of the vertices. As in the proof of Theorem 3.1, we define the sets $L(M)$ and $L_a(M)$ of labelings and admissible labelings of M. Each 2-simplex $\Delta = ABC$ of M with $A < B < C$ is provided with distinguished orientation which induces the direction from A to B on the edge AB. For any $\ell \in L(M)$, set $\ell_1^\Delta = \ell(AB)$, $\ell_2^\Delta = \ell(BC)$, and $\ell_3^\Delta = \ell(CA)$. Then

$$Z_{[c]}(M) = (\#G)^{-k_0} \sum_{\ell \in L_a(M)} \prod_{\Delta} c(\ell_1^\Delta, \ell_2^\Delta). \tag{4.3}$$

Let $\mathcal{A} = A^{(c)}$. The vector $v \in \mathcal{A}\otimes\mathcal{A}$ is computed by

$$v = (\#G)^{-1} \sum_{g \in G} c(g, g^{-1}) g \otimes g^{-1}$$

and

$$(\mathrm{id}_{\mathcal{A}} \otimes *)(v) = (* \otimes \mathrm{id}_{\mathcal{A}})(v) = (\#G)^{-1} \sum_{g \in G} g \otimes g.$$

Let $\{\mathcal{A}_f\}_f$ be a set of copies of $\mathcal{A}$ numerated by all flags f of M. With a labeling ℓ of M we associate a vector $V(\ell) \in \otimes_f \mathcal{A}_f$ as in the proof of Theorem 3.1. Then

$$V = (\#G)^{-k_1} \sum_{\ell \in L(M)} \langle\langle c, \ell \rangle\rangle V(\ell),$$

for

$$\langle\langle c, \ell \rangle\rangle = \prod_e c(\ell(e), \ell(-e)) \in F^*,$$

where e runs over all edges of M such that the distinguished orientations of the 2-simplices adjacent to e induce opposite orientations on e.

Set $U = (\#G)^{-1} T^{(3)} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to F$ and observe that

$$I_{(A^{(c)},*)} = \widetilde{T}^{(3)}(V) = (\#G)^{k_2} \widetilde{U}(V) = (\#G)^{k_2 - k_1} \sum_{\ell \in L(M)} \langle\langle c, \ell \rangle\rangle \widetilde{U}(V(\ell))$$
$$= (\#G)^{\chi(M) - k_0} \sum_{\ell \in L_a(M)} \langle\langle c, \ell \rangle\rangle \widetilde{U}(V(\ell)).$$

Here

$$\widetilde{U}(V(\ell)) = \prod_\Delta c(\ell_1^\Delta, \ell_2^\Delta) u_\Delta,$$

where $u_\Delta = c(\ell_3^\Delta, (\ell_3^\Delta)^{-1})$. If an edge e of M is adjacent to the 2-simplices Δ, Δ', then e contributes $c(\ell(e), \ell(-e)) = \pm 1$ to the product $u_\Delta u_{\Delta'}$ if the distinguished orientations of Δ, Δ' induce opposite orientations on e. Otherwise e contributes $+1$ to $u_\Delta u_{\Delta'}$. Therefore $\prod_\Delta u_\Delta = \langle\langle c, \ell \rangle\rangle$. The rest of the proof is straightforward.

Proof of Theorem 4.1. To deduce Theorem 4.1 from Theorem 4.3, we split $A^{(c)}$ as a direct product of matrix algebras. The involution $*$ on $A^{(c)}$ induces a permutation σ on the set of these algebras. The fixed points of σ bijectively correspond to the self-dual irreducible c-representations of G. The free orbits of σ give rise to $*$-subalgebras of $A^{(c)}$ of type $B = \mathrm{Mat}_d(F) \times \mathrm{Mat}_d(F)$, where d is invertible in F and the involution $*$ on B acts by $(P_1, P_2) \mapsto (P_2^{\mathrm{Tr}}, P_1^{\mathrm{Tr}})$ for $P_1, P_2 \in \mathrm{Mat}_d(F)$. A computation similar to the one in [11, Theorem 4.2] shows that $I_{(B,*)}(M) = 0$. The rest of the argument goes as in the oriented case. □

Remark 4.4. For $c = 1$, Formula (4.1) may be rewritten as

$$\#\mathrm{Hom}(\pi_1(M), G) = \#G \sum_{\rho \in \widehat{G}} \left(\frac{\varepsilon_\rho \#G}{\dim \rho} \right)^{-\chi(M)} \pmod p,$$

where $p \geq 0$ is the characteristic of F and $\widehat{G}$ is the set of irreducible linear representations of G over F considered up to linear equivalence.

Remark 4.5. Consider in more detail the case $M = P^2$. The group $\pi = \pi_1(P^2)$ is a cyclic group of order 2 so that the homomorphisms $\pi \to G$ are numerated by elements of the set $S = \{g \in G \,|\, g^2 = 1\}$. Any $g \in S$ gives rise to a (non-homogeneous) generator $[g|g]$ of the normalized bar-complex of G. This generator is a cycle modulo 2 since $\partial[g|g] = 2[g] - [g^2] = 0 \pmod 2$. For $\alpha \in H^2(G; \mathbb{Z}/2\mathbb{Z})$ and any mapping $f : P^2 \to K(G, 1)$, we have $\langle f^*(\alpha), [P^2] \rangle = \alpha([g|g])$, where $g \in G$ is the value of the induced homomorphism $f_\# : \pi \to G$ on the non-trivial element of π and $\alpha([g|g])$ is the evaluation of α on the 2-cycle $[g|g]$. Therefore

$$Z_\alpha(P^2) = (\#G)^{-1} \sum_{g \in S} (-1)^{\alpha([g|g])}.$$

If α is represented by a normalized 2-cocycle $c : G \times G \to \{\pm 1\} \approx \mathbb{Z}/2\mathbb{Z}$, then

$$(-1)^{\alpha([g|g])} = c(g, g) \quad \text{and} \quad Z_{[c]}(P^2) = (\#G)^{-1} \sum_{g \in S} c(g, g).$$

Formula (4.1) can now be rewritten as

$$\sum_{\rho\in\widehat{G}_c} \varepsilon_\rho \dim\rho = \sum_{g\in S} c(g,g) \pmod p,$$

where $p \geq 0$ is the characteristic of F. For $c = 1$, this gives

$$\sum_{\rho\in\widehat{G}} \varepsilon_\rho \dim\rho = \#S \pmod p.$$

References

[1] K.S. Brown, Cohomology of Groups, in: Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, Berlin, 1982.
[2] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Comm. Math. Phys. 129 (1990) 393–429.
[3] D. Freed, Higher algebraic structures and quantization, Comm. Math. Phys. 159 (2) (1994) 343–398.
[4] D. Freed, F. Quinn, Chern–Simons theory with finite gauge group, Comm. Math. Phys. 156 (1993) 435–472.
[5] M. Fukuma, S. Hosono, H. Kawai, Lattice topological field theory in two dimensions, Comm. Math. Phys. 161 (1) (1994) 157–175.
[6] G. Jones, Enumeration of homomorphisms and surface-coverings, Q. J. Math. 46 (2) (1995) 485–507.
[7] V. Karimipour, A. Mostafazadeh, Lattice topological field theory on nonorientable surfaces, J. Math. Phys. 38 (1997) 49–66.
[8] G. Karpilovsky, Projective Representations of Finite Groups, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 94, Marcel Dekker, Inc., New York, 1985.
[9] A.D. Mednykh, Determination of the number of nonequivalent coverings over a compact Riemann surface, Dokl. Akad. Nauk SSSR 239 (1978) 269–271. English translation: Sov. Math. Dokl. 19 (1978) 318–320.
[10] A.D. Mednykh, On the solution of the Hurwitz problem on the number of nonequivalent coverings over a compact Riemann surface, Dokl. Akad. Nauk SSSR 261 (1981) 537–542. English translation: Sov. Math. Dokl. 24 (1981) 541–545.
[11] N. Snyder, Mednykh's formula via lattice topological quantum field theories. math/0703073.
[12] V. Turaev, Homotopy field theory in dimension 2 and group-algebras. math/9910010.
[13] M. Wakui, On Dijkgraaf–Witten invariant for 3-manifolds, Osaka J. Math. 29 (4) (1992) 675–696.